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Temperature modulations in a circuit model of microwave heating

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Abstract. The problem of temperature-induced modulations on the electric field during microwave heating can be modelled as the problem of a voltage response to a periodically excited resonant circuit with a thermistor. However, even this simplified model is formulated in terms of partial differential equations. Asymptotic methods are used to reduce the problem to a phase-plane problem for the temperature profile coupled to the envelope equation for the voltage. It is shown how relaxation oscillations arise where the temperature modulates the voltage response. This strongly suggests the possibility of wave modulation in the problem of microwave heating.

Key words: asymptotics, microwaves, modulations, thermistor

1. Introduction

In microwave heating the detailed evolution of the time-dependent wave field is not usually calculated due to the fast time scales of the waves as compared to the diffusion times. However, certain models of the heating process [1] suggest that the wave envelope can modulate due to temperature effects in the case of travelling temperature fronts.

In this note we address the question of mode modulation due to temperature in terms of a simpler model suggested and studied in [2]. The evolution of the mode envelope in a cavity is modelled by an RLC circuit subject to periodic forcing. The temperature effects are accounted for by introducing a thermistor which can modulate the response due to to ohmic heating. The problem of temperature pattern formation in the thermistor is studied in [2]. In this work we address the possibility of obtaining a modulated response due to the bistable behavior in the temperature in the thermistor. We will show that relaxation oscillations are possible. These results strongly suggest the possibility of wave-mode modulation in microwave heating.

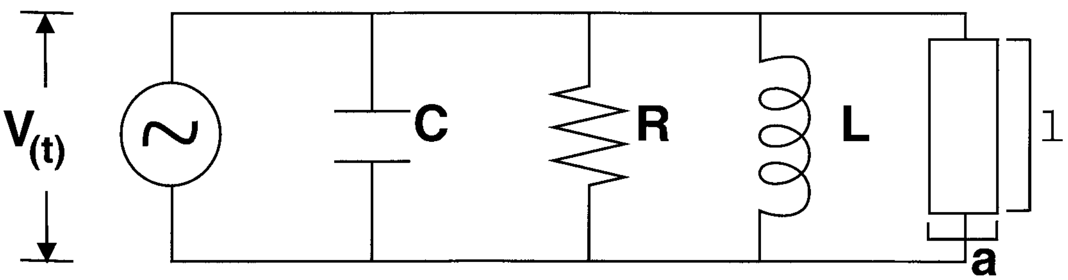

Figure 1. The RLC circuit. The thermistor is indicated by the cylinder with radius a and length l.

2. Formulation

The problem is formulated as in [2], see Figure 1, in terms of the voltage across the capacitor, resistor and inductance in the form

$$C\frac{\mathrm{d}^2V}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}I_r}{\mathrm{d}t} + \frac{V}{L} = \omega I_0 \cos \omega t, \tag{1}$$

where I_r is the current through the thermistor and I_0 is a reference value for the current source with frequency ω. The response voltage is denoted by V. The capacitance is denoted by C, the resistance by R and the inductance by L. To study the current I_r across the thermistor we assume that the thermistor is a cylinder of length l and radius a as shown in Figure 1. Since the temperature time scales involved are much larger than the wave time scales l/c_0 (c_0 the speed of light in vacuum), the electric field in the thermistor $\mathbf{E}$ will be just a time-dependent electrostatic field given in terms of a potential function as $\mathbf{E} = -\nabla\Phi$. In a problem of microwave heating the full Maxwell equations will have to be solved. This model is just an extreme approximation to the wave situation. However, as in the problem of hot-spot formation, even it the waves are fast their envelopes still evolve on the temperature time scales. For this reason this can be taken as a suggestive model for the microwave heating problem [1].

To close Equation (1) we model the current in the thermistor by

$$\mathbf{J} = \sigma(T)\mathbf{E}, \tag{2}$$

where the temperature dependence of the conductivity is the property that characterizes the thermistor. The current satisfies

$$\nabla \cdot \mathbf{J} = 0,$$

which gives, using Equation (2):

$$\nabla \cdot \sigma(T)\nabla\Phi = 0 \quad \text{in the cylinder}, \quad 0 \le z \le l, \quad 0 \le r \le a. \tag{3}$$

Equation (3) is supplemented with the no-flux boundary condition across the lateral surface $\frac{\partial\Phi}{\partial r} = 0$ at $r = a$, $0 \le z \le l$. At the top and bottom $z = 0$, $z = l$ the potential is given by $\Phi(s, y, l, t) = V(t)$, $\Phi(x, y, 0, t) = 0$.

The current through the thermistor is given by:

$$I_r(z) = \int\int_\Omega \sigma \frac{\partial\Phi}{\partial z}\mathrm{d}x\mathrm{d}y, \tag{4}$$

where Ω is the circle of radius a. Then the conservation law of Equation (3) gives, after integration:

$$\frac{\partial}{\partial z}\int\int_\Omega \sigma \frac{\partial\Phi}{\partial z}\mathrm{d}x\mathrm{d}y = 0, \tag{5}$$

and we have that $I_r(t)$ is independent of the vertical distance z along the thermistor.

To close the system we need the equation for the temperature T in the thermistor. This is the same equation as that arising in microwave heating. It includes the source $\sigma(T)|\nabla\Phi|^2$ of ohmic heating and takes the form

$$\rho c_p T_t = \nabla \cdot K(T)\nabla T + \sigma(T)|\nabla\Phi|^2. \tag{6}$$

The boundary conditions at the lateral walls, $r = a$, $0 \le z \le l$, are given by Newton's cooling law and the radiation condition which give:

$$K\frac{\partial T}{\partial r} + h_2(T - T_A) + s(T^4 - T_A^4) = 0,$$

where the ambient temperature is T_A and the conductivity is $K(T)$. The top and bottom surfaces are assumed to be insulated, *i.e.* $T_z = 0$ at $z = 0, z = l$. Equations (1), (3), (4) and (6) provide a closed system for the voltage and temperature. In principle, the potential Φ will be determined as a passive variable in terms of T and V. The initial value for the temperature is taken to be T_A. In the next section we will use the disparity of time scales involved in the problem to study the equations asymptotically and reduce them to a simple phase-plane problem.

3. Asymptotics and reduced equations

We begin by scaling the dependent and independent nondimensional variables in the form:

$$\begin{aligned} v &= V/V_0, \quad \varphi = \Phi/V_0, \quad u = (T - T_A)/T_, \\ x' &= x/a, \quad y' = y/a, \quad z' = z/l, \quad t' = \omega t, \end{aligned}$$

and the material properties are scaled as:

$$k(u) = K(T)/K_0, \quad f(u) = \sigma(T)/\sigma_0, \quad i = \frac{I_r}{I_1}, \tag{7}$$

where $R_0 = \sqrt{L/C}$, $\omega_0 = 1/\sqrt{LC}$, $\sigma_0 = \sigma(T_A)$, $K_0 = K(T_A)$ are the values of the material parameters at the ambient temperature and $V_0 = RI_0$ and $I_1 = \sigma_0 a^2 V_0/l$; I_1 is the current in the thermistor at room temperature. With these scalings we have, after dropping the primes, that the voltage satisfies:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \nu\beta\frac{\mathrm{d}v}{\mathrm{d}t} + \nu^2 v + \nu\gamma\beta\frac{\mathrm{d}i}{\mathrm{d}t} = \nu\beta \cos t, \quad v(0) = v_0'(0) = 0. \tag{8}$$

In Equation (8) $\nu = \omega_0/\omega$, with $\beta = R_0/R$, and $\gamma = \sigma_0 a^2 R/l$.

In nondimensional variables the electrostatic potential satisfies the equation:

$$\frac{\partial}{\partial x}\left(f(u)\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(f(u)\frac{\partial}{\partial y}\varphi\right) + \epsilon^2\frac{\partial}{\partial z}\left(f(u)\frac{\partial\varphi}{\partial z}\right) = 0 \quad \text{for} \quad 0 \le r \le 1, 0 \le z \le 1, \tag{9}$$

$$\begin{aligned} &\varphi_r = 0 \quad \text{at} \quad r = 1, \\ &\varphi(s, y, 1, t) = v(t), \\ &\varphi(x, y, 0, t) = 0. \end{aligned} \tag{9a}$$

The parameter $\epsilon = a/l$ is the slenderness parameter. The non-dimensional current i is given by:

$$i = \gamma\int\int_\Omega f(u)\frac{\partial\varphi}{\partial z}\mathrm{d}x\mathrm{d}y. \tag{10}$$

In the same way the temperature equation in the scaled variables becomes:

$$\frac{\partial u}{\partial t} = \delta \left\{ L_\perp(k)u + \epsilon^2 L_z(k)u + pf(u)\left(|\nabla_\perp \varphi|^2 \left(\frac{\partial \varphi}{\partial z}\right)^2\right)\right\}, \tag{11}$$

where

$$L_\perp(k)u = \nabla_\perp \cdot (k(u)\nabla_\perp)u, \qquad L_z(k)u = \frac{\partial}{\partial z}\left(k(u)\frac{\partial u}{\partial z}\right).$$

$\delta = (K_0/\rho c_p a^2)/\omega$ is the ratio of the source period to the diffusion time scale. The parameter $p = \sigma_0 V_0^2/K_0 T_0$ is the ratio between the electric energy and the thermal energy. The operator $\nabla_\perp$ is the gradient in the transverse variables x, y. The boundary conditions are scaled as:

$$\begin{aligned} k\frac{\partial u}{\partial z} &= 0 \quad \text{at} \quad z = 0, 1 \quad 0 \le r \le 1. \\ k\frac{\partial u}{\partial z} &+ B\epsilon^2 g(u) = 0 \quad \text{at} \quad r = 1, 0 \le z \le 1. \end{aligned} \tag{12}$$

where B is the Biot number hl/k_0, and $g(u) = u + R((u+1)^4 - 1)$, $R = s/h_2$.

To further reduce the equations we assume that the circuit described in (8) is resonant, *i.e.* ν is set equal to 1; β is assumed to be small. The voltage equation (8) simplifies to:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \beta v_t + v + \beta\gamma\frac{\mathrm{d}}{\mathrm{d}t}\int\int_\Omega f(u)\frac{\partial \varphi}{\partial z}\mathrm{d}x\mathrm{d}y = \beta \cos(t + t_0). \tag{13}$$

The phase to is introduced for convenience. Up to this stage the problem is the same as the one studied in [2]. However, instead of studying, as in [2], steady stable time-periodic solutions producing steady spatially inhomogeneous temperature patterns, we focus on the possibility of producing a periodic solution, of relaxation type in the temperature. This regime is only accessible in the asymptotics when the forcing is comparable to the damping. We thus take $\beta \ll 1$. With this the solution of (13) is taken in the farm of a modulated oscillation on the slow time γt. This is

$$v = A(\beta t)\mathrm{e}^{\mathrm{i}t} + c.c. \tag{14}$$

where $c.c.$ denotes the complex conjugate. To find the modulation equation for A is necessary to consider the equation for the temperature. Since $\delta \ll 1$, the temperature evolves slowly relative to the oscillation period. We thus average the temperature equation, taking $u = u(t, x, y, z, \tau)$ with $\tau = \delta t$. In this scaling the usual procedure [2] gives.

$$\frac{\partial u}{\partial \tau} = L_\perp(k(u))u + \epsilon^2 L_z(k(u))u + \lim_{T\to\infty}\frac{1}{T}\int_0^T pf(u)\left(|\nabla_\perp\varphi|^2 + \epsilon^2\left(\frac{\partial\varphi}{\partial z}\right)^2\right)\mathrm{d}t. \tag{15}$$

Using the representation (14) in (15) to evaluate the limit, we obtain:

$$\frac{\partial u}{\partial \tau} = L_\perp(k(u))u + \epsilon^2 L_z(k(u))u + pf(u)|A|^2\left(|\nabla_\perp\Psi|^2 + \epsilon(\Psi_z)^2\right). \tag{16}$$

In Equation (16) the function Ψ satisfies the scaled problem (9) with condition $\Psi(x, y, 1, t) = 1$ instead of (9a).

The modulation equation for A to leading order in β takes the form:

$$2\mathrm{i}\dot{A} + \mathrm{i}A + \mathrm{i}A\gamma\int\int_\Omega f(u)\frac{\partial\Psi}{\partial z}\mathrm{d}x\mathrm{d}y = \mathrm{e}^{\mathrm{i}t_0}. \tag{17}$$

Thus Equations (9), (16) and (17) provide a simplification of the original equations.

A further reduction can be achieved if we use the fact that $\epsilon \ll 1$. In this case we take:

$$\begin{aligned} u &= u^0 + \epsilon^2 u^1 + \dots \\ \Psi &= \Psi^0 + \epsilon^2 \Psi^1 + \dots \end{aligned}$$

Then to leading order the potential equation (9) becomes:

$$\begin{aligned} \nabla_\perp \cdot f(u^0)\nabla\Psi^0 &= 0, \\ \Psi^0_r &= 0. \end{aligned} \tag{18}$$

Since $f(u^0) > 0$, Equation (18) gives that Ψ_0 is independent of (x, y). To the next order we determine the z-dependence of Ψ_0 from:

$$\begin{aligned} \frac{\partial}{\partial z} f(u^0) \frac{\partial}{\partial z}\Psi^0 &= 0, \\ \Psi^0(1) = 1, \quad \Psi^0(0) &= 0. \end{aligned} \tag{19}$$

This is readily integrated to obtain

$$\Psi^0(z) = \int_0^z \frac{1}{f(u^0)} \mathrm{d}z' \Big/ \int_0^1 \frac{1}{f(u^0)} \mathrm{d}z.$$

The last equation to be simplified is the temperature equation. For small ϵ the equation has been simplified by Kriegsmann [4], using again averaging in the spatial variables. Introducing the slower time $\tilde{t} = \epsilon^2 \tau$ we have that the averaged temperature on the cross section denoted again by u, which depends only on the z variable, satisfies the one-dimensional functional equation.

$$\frac{\partial u^0}{\partial \tilde{t}} = \frac{\partial}{\partial z}\left(k(u^0)\frac{\partial u^0}{\partial z}\right) + \frac{1}{\pi}\left(-Bg(u^0) + p|A|^2 \frac{1}{f(u^0)} \left(\int_0^1 \frac{\partial 1}{f(u^0)} \mathrm{d}z\right)^{-2}\right) \tag{20}$$

with insulated boundary conditions at $z = 0$, and $z = 1$.

With this simplification Equations (17), (19) and (20) control the evolution of the modulation coupled to the temperature profile. Equation (17) can be further simplified by use of the explicit form for Ψ^0. We also choose to obtain for A a real modulation equation the phase $t_0 = \pi/2$. We thus have

$$\begin{aligned} \dot{A} &+ (1 + \gamma Z(A^2))A = 1, \\ Z(A^2) &= \frac{\pi}{2}\left(\int_0^1 \frac{\mathrm{d}z}{f(u^0)}\right)^{-1} \end{aligned} \tag{21}$$

and $|A|^2$ is replaced by just A^2 in (20). These equations are the modulation equation for both voltage amplitude and temperature. We have assumed that the parameter $\gamma \sim O(\delta\epsilon^2)$ for (20) and (21) to be on the same time scale.

Equation (20) for given A^2 has been extensively studied by Kriegsmann [2], [4] in the context of spatially dependent solutions. In those studies it was assumed that the temperature did not feed back on the voltage. Eventually the voltage reached a steady state and the temperature a stable spatial pattern. On the other hand, the system (20) and (21) will be shown in the next section to exhibit periodic relaxation oscillation type solutions.

4. Periodic solutions

We consider solutions of (20) and (21) that are z-independent. In this case (20) and (21) become just ordinary differential equations. If we denote by θ the spatially homogeneous temperature, they take the form

$$\begin{aligned} \dot{A} &+ (1 + \frac{\pi}{2}\gamma f(\theta))A = 1, \\ \dot{\theta} &= \frac{1}{\pi}\{pA^2 f(\theta) - Bf(\theta)\}. \end{aligned} \tag{22}$$

To close the system the function $f(\theta)$ which charaterizes the response of the thermistor and depends on the material is taken to be

$$f(\theta) = (1 + c_1)\mathrm{e}^{-c_2/\theta}, \tag{23}$$

which saturates at $1 + c_1$ as $\theta \to \infty$. This function is also appropriate for other models of microwave heating [3].

To study the phase plane of (22) we observe that the nullclines are given by

$$(1 + \frac{\pi}{2}\gamma f(\theta))A = 1, \tag{24a}$$

$$pA^2 f(\theta) - Bg(\theta) = 0. \tag{24b}$$

The graph of the curve (22b) is the S-shaped curve shown in Figure 2. As expected in these problems the top and bottom branches are stable, while the middle branch is unstable. The curve (22a) given by

$$A = \frac{2}{2 + \pi\gamma f(\theta)}$$

intersects the S-shaped curve in the middle branch providing the fixed point. (This intersection always occurs since A is a decreasing function of θ and the amplitude γ is at our disposal). The point P of intersection is thus unstable. The orbits move away from P and are acttracted to the stable branches. This is the typical situation for the existence of a limit cycle [5, Chapter 16]. In fact, on the boundary of the region Ω shown in Figure 2 the vector field always points inwards Ω, as can be readily seen from Equations (22). Thus, since the fixed point P is unstable, we have by the Poincaré-Bendixon Theory a limit cycle as the attracting orbit. Moreover, if $p \sim B$, and we take B large, the oscillation will be a relaxation oscillation. The orbit travels along the lower branch up to the fold, it jumps to A_1 rapidly and the process is repeated.

In the case of relaxation we can construct a very elementary model with an exact solution which displays the desired relaxation oscillation. In fact, assume $\theta_0(A^2)$ to be the parametrization of the lower branch and $\theta_1(A^2)$ the corresponding one for the upper branch of the S-shaped curves. On these branches A satisfies a simple ordinary differential equation:

$$\dot{A} + (1 + \gamma Z(\theta_{01}(A^2)))A = 1,$$

together with the continuity condition of A at the jumps (or folds) at A_0, and A_1; see Figure 2. We further assume to obtain an exact solution, namely $1 + \gamma Z(\theta_0, (A)^2) = r_0$. The damping r_1 on the upper branch satisfies $r_1 > r_0$, where r_0 is the damping on the lower branch.

The solution is readily constructed by matching at A_0 and A_1 the solutions of the slow linear problems. It is easy to see that, when the initial condition is $a(0) = q$, the solution is an increasing exponential given by

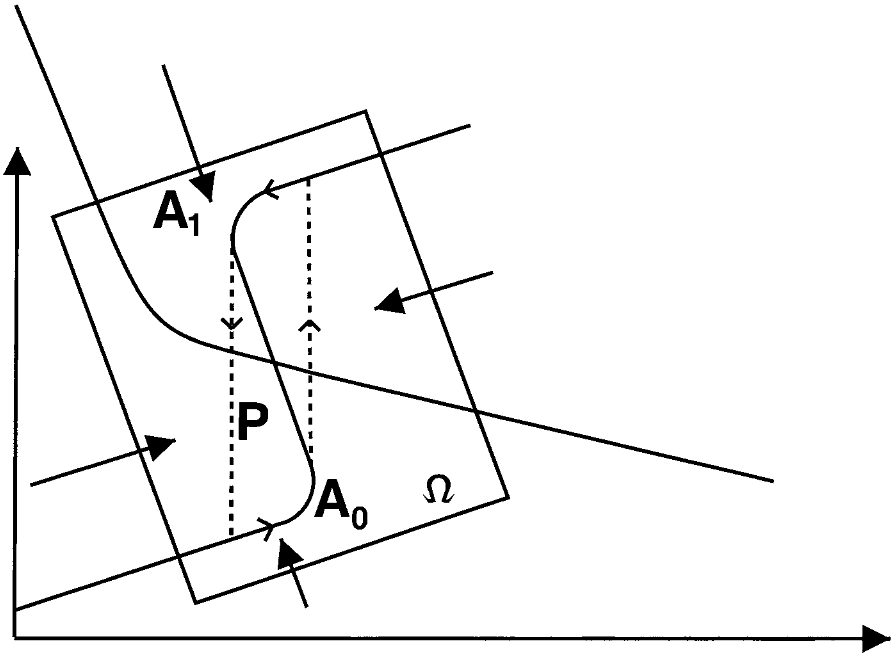

Figure 2. The phase plane for the envelope and temperature. The dotted orbit is the relaxation oscillation.

$$a(t) = p/r_0(1 - e^{-r_0 t}) + q e^{-r_0 t}, \quad 0 \le t \le -\frac{1}{r_0} \log\left(\frac{p/r\_A - 1}{p/r_0 - q}\right) = t_1.$$

Using the jump conditions, we obtain the value of t_2 and the corresponding solution in the form:

$$a(t) = (A_1 - p/r_1)e^{-r_1(t-t_1)+r/r_1}, \quad t_1 \le t \le t_2 = \log\left(\frac{p/r_0 - q}{p/r_0 - A_1}\right)^{\frac{1}{r_0}} \left(\frac{A_1 - p/r_0}{A_0 - p/r_1}\right)^{\frac{1}{r_1}}.$$

Finally, matching periodically at q, we find the time t_3, which is the period of the oscillation. In this interval the solution takes the form:

$$a(t) = A_0 - p(r_0)e^{-r_1(t-t_2)} + p/r_0 \text{ for } t_2 \le t \le t_3 = \log\left(\frac{p/r_0 - A_0}{p/r_0 - A_1}\right)^{1/r_0}\left(\frac{A_1 - p/r_1}{A_0 - p/r_1}\right)^{\frac{1}{r_1}}.$$

Notice that the period $T = t_3$ is independent of the initial value q. Thus, all possible initial conditions give the same periodic solution translated in time. This simple solution clearly shows the periodic modulation caused on the voltage response due to the bistable nature of the temperature in the thermistor. In general, the details of the solution will have to be calculated numerically. With this analysis, the solution displays the mechanism of relaxation in simple form.

5. Conclusions

We have shown the presence of periodic modulations, due to the bistable temperature behavior of the thermistor for a simple RLC circuit. The periodic modulation is sustained by the relaxation oscillation in the temperature. This modulation occurs on a slow scale as compared with the natural period of oscillation of the input voltage.

These results strongly suggest the possibility of modulating the wave-field envelope in a microwave cavity due to the bistable temperature response. Instead of Laplace's equation, we will have Maxwell's equations. The envelope of the waves will be modulated by the bistable response. In fact, for an externally excited dominant cavity mode, the envelope equation for just time-dependent modulations will be exactly given by (21). However, spatial modulations will be also induced and will have to be taken into account.

Finally, more complex time patterns are likely to arise when the responses in temperature include spatial patterns. In this case we expect complicated responses when various patterns can be excited by the supplied voltage; also a combination of wave modulation and pattern switching may produce a great variety of responses. Some of these possibilities are currently under study.

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